

On a new integrable discretization of the derivative nonlinear Schrödinger (Chen–Lee–Liu) equation

Takayuki TSUCHIDA

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Abstract

We propose a general integrable lattice system involving some free parameters, which contains known integrable lattice systems such as the Ablowitz–Ladik discretization of the nonlinear Schrödinger (NLS) equation as special cases. With a suitable choice of the parameters, it provides a new integrable space-discretization of the derivative NLS equation known as the Chen–Lee–Liu equation. Analogously to the continuous case, the space-discrete Chen–Lee–Liu system possesses a Lax pair and admits a complex conjugation reduction between the two dependent variables. Thus, we obtain a proper space-discretization of the Chen–Lee–Liu equation defined on the three lattice sites $n - 1$, n , $n + 1$ for the first time. Considering a negative flow of the discrete Chen–Lee–Liu hierarchy, we obtain a proper discretization of the massive Thirring model in light-cone coordinates. Multicomponent generalizations of the obtained discrete equations are straightforward because the performed computations are valid for the general case where the dependent variables are vector- or matrix-valued.

1 Introduction

There exists no systematic method of obtaining a proper discretization for a given integrable partial differential equation (PDE). For some scalar PDEs such as the sine-Gordon equation $u_{xt} = \sin u$ [1–3], elementary auto-Bäcklund transformations and the associated nonlinear superposition formula based on Bianchi’s permutability theorem can provide proper discretizations of the original continuous equations. However, this idea does not apply directly to complex-valued PDEs involving the operation of complex conjugation, such as the nonlinear Schrödinger (NLS) equation $iq_t + q_{xx} \pm 2q^2q^* = 0$, which can be naturally obtained from two coupled PDEs by imposing a complex conjugation reduction on the two dependent variables [4]. This is because elementary auto-Bäcklund transformations do not, in general, maintain the complex conjugation reduction [5–8]. One can consider more elaborate auto-Bäcklund transformations by composing two or more elementary auto-Bäcklund transformations so that the net result can maintain the complex conjugation reduction between the two dependent variables. However, such composite auto-Bäcklund transformations generally involve an indefinite integral [9] or a square-root function with an indefinite sign [10, 11] (also see (7.18) in [12] or (4.17) in [13]), so they do not directly provide proper discretizations of the original PDEs, which can be written in local form and define the time evolution uniquely in the discrete setting.

Because proper discretizations of integrable PDEs involving the operation of complex conjugation cannot be derived in a systematic manner, it is more productive and instructive to obtain such discretizations on a case-by-case consideration. In this paper, we propose a new proper space-discretization of an integrable derivative NLS equation $iq_t + q_{xx} + iqq^*q_x = 0$, usually referred to as the Chen–Lee–Liu equation [14] (also see [15]). Actually, we already obtained an integrable space-discretization of the Chen–Lee–Liu equation in our previous paper [16], but it depends on five lattice sites and is rather complicated. For the Chen–Lee–Liu system in the nonreduced form, i.e., two coupled PDEs for q and q^* wherein q and q^* are independent functions not related by a complex conjugation, a fully discrete analog was derived in [8] (see also the continuous-time flows in [16–20]); a correspondence between a fully discrete system and continuous-time flows is briefly described in [21].

We can obtain a proper space-discrete Chen–Lee–Liu equation defined on three lattice sites $n - 1, n, n + 1$ by constructing its Lax-pair representation. As the spatial part of the Lax-pair representation, we consider a new discrete spectral problem implied by a binary Bäcklund–Darboux transformation for the continuous Chen–Lee–Liu equation or, equivalently, that for the Ablowitz–Ladik lattice (an integrable discrete NLS equation [22, 23]) pro-

posed in [24, 25]. Note that the Chen–Lee–Liu hierarchy and the Ablowitz–Ladik hierarchy are different facets of the same object and in a sense equivalent [26, 27] (also see [28, 29]). In contrast to the usual formulation of the Bäcklund–Darboux transformations [5–7, 30–32], we do not express two unknown functions appearing in the Darboux matrix explicitly in terms of linear eigenfunctions of the original spectral problem. Rather, we consider them as new dependent variables in the discrete spectral problem, which can be related to each other through a complex conjugation reduction. Then, we associate the discrete spectral problem with a suitable time-evolution equation, which comprises the Lax-pair representation; the compatibility condition provides a rather general lattice system involving some arbitrary parameters. Depending on the choice of the parameters, it provides a proper space-discretization of the Chen–Lee–Liu equation, as well as other integrable lattice systems such as the Ablowitz–Ladik lattice [22, 23].

This paper is organized as follows. In section 2, we introduce a discrete spectral problem and associate it with a suitable isospectral time-evolution equation. Then, the compatibility condition for this Lax-pair representation provides a new integrable lattice system; with a suitable choice of the parameters therein, we obtain a proper space-discretization of the Chen–Lee–Liu equation. In section 3, we change the time part of the Lax-pair representation to obtain a proper discretization of the massive Thirring model in light-cone (or characteristic) coordinates [33–35]. Section 4 is devoted to concluding remarks. Throughout the paper, we perform the computations for the most general case where the dependent variables are (rectangular) matrix-valued, so multicomponent generalizations of the obtained equations are straightforward.

2 Space-discrete Chen–Lee–Liu equation

In this section, we propose a new integrable lattice system, which contains a proper space-discretization of the Chen–Lee–Liu equation as a special case, through a Lax-pair representation. In subsection 2.1, we introduce a new discrete spectral problem, which gives the spatial part of the Lax pair. In subsection 2.2, we associate it with an isospectral time-evolution equation, which is the temporal part of the Lax pair.

2.1 Bäcklund–Darboux transformation as a discrete spectral problem

The Ablowitz–Ladik spectral problem [22, 23] can be generalized to a block-matrix form [36] (also see [16, 37] and references therein) as given by

$$\begin{bmatrix} \Psi_{1,m+1} \\ \Psi_{2,m+1} \end{bmatrix} = \begin{bmatrix} \zeta I & Q_m \\ R_m & \frac{1}{\zeta} I \end{bmatrix} \begin{bmatrix} \Psi_{1,m} \\ \Psi_{2,m} \end{bmatrix}. \quad (2.1)$$

Here, ζ is a constant spectral parameter and Q_m and R_m are, respectively, $l_1 \times l_2$ and $l_2 \times l_1$ (generally rectangular) matrices. Thus, the square matrix above is partitioned as an $(l_1 + l_2) \times (l_1 + l_2)$ block matrix; for notational brevity, we omit the index of each unit matrix I to indicate its size.

Among an infinite set of isospectral time-evolution equations compatible with the Ablowitz–Ladik spectral problem (2.1) (cf. [38, 39]), the most fundamental ones are

$$\begin{bmatrix} \Psi_{1,m} \\ \Psi_{2,m} \end{bmatrix}_{t_1} = \begin{bmatrix} \zeta^2 I - Q_m R_{m-1} & \zeta Q_m \\ \zeta R_{m-1} & O \end{bmatrix} \begin{bmatrix} \Psi_{1,m} \\ \Psi_{2,m} \end{bmatrix} \quad (2.2)$$

and

$$\begin{bmatrix} \Psi_{1,m} \\ \Psi_{2,m} \end{bmatrix}_{t_{-1}} = \begin{bmatrix} O & \frac{1}{\zeta} Q_{m-1} \\ \frac{1}{\zeta} R_m & \frac{1}{\zeta^2} I - R_m Q_{m-1} \end{bmatrix} \begin{bmatrix} \Psi_{1,m} \\ \Psi_{2,m} \end{bmatrix}. \quad (2.3)$$

Here, the subscript t_j denotes time differentiation and the symbol italic O represents a zero matrix. The corresponding equations of motion are derived from the compatibility condition for the overdetermined linear system, (2.1) and (2.2) or (2.3), i.e.,

$$\begin{cases} Q_{m,t_1} - Q_{m+1} + Q_{m+1} R_m Q_m = O, \\ R_{m,t_1} + R_{m-1} - R_m Q_m R_{m-1} = O, \end{cases} \quad (2.4)$$

and

$$\begin{cases} Q_{m,t_{-1}} + Q_{m-1} - Q_m R_m Q_{m-1} = O, \\ R_{m,t_{-1}} - R_{m+1} + R_{m+1} Q_m R_m = O. \end{cases} \quad (2.5)$$

These are the two elementary flows of the Ablowitz–Ladik hierarchy and a suitable linear combination of them together with the trivial zeroth flow provides the Ablowitz–Ladik discretization of the NLS system [22, 40]. Note that the use of O , instead of 0, on the right-hand side of the equations implies that the dependent variables are matrix-valued.

A crucial fact noticed by Vekslerchik [26, 27] (also see Barashenkov–Getmanov [28, 29]) is that the Ablowitz–Ladik hierarchy is in a sense equivalent to a derivative NLS hierarchy called the Chen–Lee–Liu hierarchy [14]. This is clear from the observation that the isospectral equation (2.2) with $t_1 \rightarrow x$ (or (2.3) with $t_{-1} \rightarrow x$) has essentially the same form as the spectral problem associated with the continuous Chen–Lee–Liu hierarchy [41–43] (see [44, 45] for the matrix case); note that this spectral problem can be rewritten in a traceless form using a simple gauge transformation [28]. Actually, the Chen–Lee–Liu spectral problem was first presented for the massive Thirring model in light-cone (or characteristic) coordinates [33–35] (also see [46, 47] for the case of laboratory coordinates), which is the first negative flow of the Chen–Lee–Liu hierarchy [44, 48]; the compatibility condition for the overdetermined linear system, (2.2) and (2.3) for any fixed value of m , indeed provides the equations of motion for the massive Thirring model ((2.4) for Q_{m-1,t_1} and R_{m,t_1} and (2.5) for $Q_{m,t_{-1}}$ and $R_{m-1,t_{-1}}$). Conversely, we can regard the Ablowitz–Ladik spectral problem (2.1) as a Bäcklund–Darboux transformation for the continuous Chen–Lee–Liu hierarchy [28, 29]; then the elementary flow (2.4) (or (2.5)) of the Ablowitz–Ladik hierarchy can be considered as an auto-Bäcklund transformation for the continuous Chen–Lee–Liu hierarchy. These items of information imply that a suitably chosen Bäcklund–Darboux transformation for the Ablowitz–Ladik hierarchy can provide a proper discretization of the Chen–Lee–Liu spectral problem.

In fact, Bäcklund–Darboux transformations for the Ablowitz–Ladik hierarchy have already been considered in a number of papers (see, *e.g.*, [24, 32, 49, 50]); those results are closely related to integrable time-discretizations of the flows of the Ablowitz–Ladik hierarchy [25, 27, 38, 40, 51, 52]. Noting that the Ablowitz–Ladik hierarchy is invariant under the rescaling $Q_m \rightarrow kQ_m$, $R_m \rightarrow k^{-1}R_m$, we can introduce a slightly generalized version of the binary Bäcklund–Darboux transformation in the form (cf. [24, 25, 53]):

$$\begin{bmatrix} \tilde{\Psi}_{1,m} \\ \tilde{\Psi}_{2,m} \end{bmatrix} = \left\{ \begin{bmatrix} \left(\alpha\zeta + \frac{\delta}{\zeta}\right) I & \\ & \left(\gamma\zeta + \frac{\beta}{\zeta}\right) I \end{bmatrix} + (\alpha\beta - \gamma\delta) \begin{bmatrix} \gamma\zeta I & u_m \\ v_m & \frac{\delta}{\zeta} I \end{bmatrix}^{-1} \right\} \begin{bmatrix} \Psi_{1,m} \\ \Psi_{2,m} \end{bmatrix}. \quad (2.6)$$

Here, α , β , γ and δ are arbitrary parameters and u_m and v_m are intermediate potentials that can be expressed in terms of linear eigenfunctions of the Ablowitz–Ladik spectral problem (2.1) at special values of the spectral parameter ζ ; however, this information is not necessary for the purpose of this paper and we will treat u_m and v_m as new independent unknowns. Alternatively, we can consider (2.6) as a generalized binary Bäcklund–Darboux transformation for the isospectral problem (2.2) or (2.3) associated with

the Chen–Lee–Liu hierarchy. The binary Bäcklund–Darboux transformation for the Chen–Lee–Liu hierarchy (or, more specifically, the massive Thirring model) was apparently first considered by P. I. Holod (also written as Golod, preprint in Russian, Kiev, 1978) in a somewhat preliminary form; the result can be found in more accessible papers [54–56] (see [28, 29, 57, 58] for further developments).

Because the Ablowitz–Ladik hierarchy is invariant under a space translation $m \rightarrow m + l$, Bäcklund–Darboux transformations can always be combined with a space translation; for instance, using (2.6) and (2.1), we can compute

$$\begin{bmatrix} \tilde{\Psi}_{1,m+1} \\ \tilde{\Psi}_{2,m+1} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \tilde{\Psi}_{1,m-1} \\ \tilde{\Psi}_{2,m-1} \end{bmatrix}$$

and consider it as a new Bäcklund–Darboux transformation, which has the same ζ -dependence as that usually employed in the time-discretizations of the Ablowitz–Ladik flows [23, 40, 51, 52]. Note also that an overall factor of (2.6) is nonessential, so we can multiply the right-hand side of (2.6) by ζ or $1/\zeta$ to obtain more familiar ζ -dependence that appears in the time-discretizations of the Ablowitz–Ladik flows.

Now, we freeze the lattice index m for the Ablowitz–Ladik hierarchy and reinterpret the Bäcklund–Darboux transformation (2.6) as defining a new discrete spectral problem:

$$\begin{bmatrix} \Psi_{1,n+1} \\ \Psi_{2,n+1} \end{bmatrix} = L_n \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}, \quad (2.7)$$

where n can be intuitively understood as the number of iterations of the

Bäcklund–Darboux transformation. Here, the Lax matrix L_n is given by

$$L_n = \begin{bmatrix} \left(\alpha\zeta + \frac{\delta}{\zeta}\right) I & \\ & \left(\gamma\zeta + \frac{\beta}{\zeta}\right) I \end{bmatrix} + (\alpha\beta - \gamma\delta) \begin{bmatrix} \gamma\zeta I & u_n \\ v_n & \frac{\delta}{\zeta} I \end{bmatrix}^{-1} \quad (2.8a)$$

$$= \begin{bmatrix} \left(\alpha\zeta + \frac{\delta}{\zeta}\right) I & \\ & \left(\gamma\zeta + \frac{\beta}{\zeta}\right) I \end{bmatrix} + (\alpha\beta - \gamma\delta) \begin{bmatrix} \frac{\delta}{\zeta}(\gamma\delta I - u_n v_n)^{-1} & -(\gamma\delta I - u_n v_n)^{-1} u_n \\ -(\gamma\delta I - v_n u_n)^{-1} v_n & \gamma\zeta(\gamma\delta I - v_n u_n)^{-1} \end{bmatrix} \quad (2.8b)$$

$$= \begin{bmatrix} \alpha\zeta I & u_n \\ v_n & \frac{\beta}{\zeta} I \end{bmatrix} \begin{bmatrix} \left(\gamma\zeta + \frac{\beta}{\zeta}\right) I & \\ & \left(\alpha\zeta + \frac{\delta}{\zeta}\right) I \end{bmatrix} \begin{bmatrix} \gamma\zeta I & u_n \\ v_n & \frac{\delta}{\zeta} I \end{bmatrix}^{-1} \quad (2.8c)$$

$$= \begin{bmatrix} \gamma\zeta I & u_n \\ v_n & \frac{\delta}{\zeta} I \end{bmatrix}^{-1} \begin{bmatrix} \left(\gamma\zeta + \frac{\beta}{\zeta}\right) I & \\ & \left(\alpha\zeta + \frac{\delta}{\zeta}\right) I \end{bmatrix} \begin{bmatrix} \alpha\zeta I & u_n \\ v_n & \frac{\beta}{\zeta} I \end{bmatrix}. \quad (2.8d)$$

The parameters α , β , γ and δ can be varied at each application of the Bäcklund–Darboux transformation, so they can be arbitrary functions of the discrete independent variable n ; however, for simplicity we consider them as constants. Each of the four equivalent expressions in (2.8) has its own advantages.

2.2 Isospectral time-evolution equation

To compose a Lax pair, we associate (2.7) with a suitable isospectral time-evolution equation,

$$\begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}_t = M_n \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}. \quad (2.9)$$

The compatibility condition for the overdetermined linear system, (2.7) and (2.9), is given by (a space-discrete version of) the zero-curvature equation [22, 23, 39, 59]

$$L_{n,t} + L_n M_n - M_{n+1} L_n = O, \quad (2.10)$$

where L_n and M_n comprise the Lax pair. For the Lax matrix L_n in (2.8), the first expression (2.8a) allows us to compute $L_{n,t}$ in such a way that the time-evolution equations for u_n and v_n can be obtained from (2.10) explicitly; note that $(X^{-1})_t = -X^{-1}X_tX^{-1}$ for a square matrix X .

To obtain a proper space-discrete analog of the Chen–Lee–Liu system, we consider a temporal Lax matrix M_n that goes well with the factorized form

(2.8c) or (2.8d) of the spatial Lax matrix L_n . Thus, a natural ansatz for M_n is

$$M_n = (\alpha\beta - \gamma\delta) \begin{bmatrix} \frac{\beta}{\zeta}I & -u_n \\ -v_n & \alpha\zeta I \end{bmatrix} \begin{bmatrix} \frac{\gamma\zeta}{\gamma\zeta + \frac{\beta}{\zeta}}F_n & \\ & \frac{\frac{\delta}{\zeta}}{\alpha\zeta + \frac{\delta}{\zeta}}G_n \end{bmatrix} \begin{bmatrix} \frac{\beta}{\zeta}I & -u_{n-1} \\ -v_{n-1} & \alpha\zeta I \end{bmatrix} + \begin{bmatrix} cI & \\ & dI \end{bmatrix}, \quad (2.11)$$

where F_n and G_n are square matrices, and c and d are arbitrary constants introduced for adding the trivial zeroth flow. The condition $\alpha\beta - \gamma\delta \neq 0$ is assumed for the Lax pair to be nontrivial. Substituting (2.8) and (2.11) into (2.10), we obtain recursion relations for determining F_n and G_n ; they are satisfied by setting

$$F_n = a(\beta\gamma I - u_{n-1}v_n)^{-1}, \quad G_n = b(\alpha\delta I - v_{n-1}u_n)^{-1}, \quad (2.12)$$

where a and b are arbitrary constants. We can also consider the more general case where a and b , as well as c and d in (2.11), are arbitrary functions of time t , but we do not discuss it in this paper. Then, (2.10) for (2.8) and (2.11) with (2.12) provides an evolutionary lattice system:

$$\begin{cases} u_{n,t} - a\gamma(\alpha\beta I - u_n v_n)(\beta\gamma I - u_{n-1}v_n)^{-1}(\beta u_n - \delta u_{n-1}) \\ \quad - b\delta(\gamma u_{n+1} - \alpha u_n)(\alpha\delta I - v_n u_{n+1})^{-1}(\alpha\beta I - v_n u_n) - (c - d)u_n = O, \\ v_{n,t} - b\delta(\alpha\beta I - v_n u_n)(\alpha\delta I - v_{n-1}u_n)^{-1}(\alpha v_n - \gamma v_{n-1}) \\ \quad - a\gamma(\delta v_{n+1} - \beta v_n)(\beta\gamma I - u_n v_{n+1})^{-1}(\alpha\beta I - u_n v_n) + (c - d)v_n = O. \end{cases} \quad (2.13)$$

In the case of $\beta = \alpha^*$, $\delta = \gamma^*$, $b = a^*$ and $d = c^*$, system (2.13) admits the complex conjugation reduction $v_n = \sigma u_n^*$ with a real constant σ . To consider a Hermitian conjugation reduction between u_n and v_n , we need to rewrite (2.13). By setting $c - d = -a(\alpha\beta - \gamma\delta)$ and using the identities

$$\begin{aligned} & \gamma(\alpha\beta I - u_n v_n)(\beta\gamma I - u_{n-1}v_n)^{-1}(\beta u_n - \delta u_{n-1}) - (\alpha\beta - \gamma\delta)u_n \\ &= \delta(\gamma u_n - \alpha u_{n-1}) - (\gamma u_n - \alpha u_{n-1})v_n(\beta\gamma I - u_{n-1}v_n)^{-1}(\beta u_n - \delta u_{n-1}) \\ &= \beta(\gamma u_n - \alpha u_{n-1})(\beta\gamma I - v_n u_{n-1})^{-1}(\gamma\delta I - v_n u_n), \\ & \gamma(\delta v_{n+1} - \beta v_n)(\beta\gamma I - u_n v_{n+1})^{-1}(\alpha\beta I - u_n v_n) + (\alpha\beta - \gamma\delta)v_n \\ &= \delta(\alpha v_{n+1} - \gamma v_n) + (\delta v_{n+1} - \beta v_n)(\beta\gamma I - u_n v_{n+1})^{-1}u_n(\alpha v_{n+1} - \gamma v_n) \\ &= \beta(\gamma\delta I - v_n u_n)(\beta\gamma I - v_{n+1}u_n)^{-1}(\alpha v_{n+1} - \gamma v_n), \end{aligned}$$

we can rewrite (2.13) as

$$\begin{cases} u_{n,t} - a\beta(\gamma u_n - \alpha u_{n-1})(\beta\gamma I - v_n u_{n-1})^{-1}(\gamma\delta I - v_n u_n) \\ \quad - b\delta(\gamma u_{n+1} - \alpha u_n)(\alpha\delta I - v_n u_{n+1})^{-1}(\alpha\beta I - v_n u_n) = O, \\ v_{n,t} - b\delta(\alpha\beta I - v_n u_n)(\alpha\delta I - v_{n-1} u_n)^{-1}(\alpha v_n - \gamma v_{n-1}) \\ \quad - a\beta(\gamma\delta I - v_n u_n)(\beta\gamma I - v_{n+1} u_n)^{-1}(\alpha v_{n+1} - \gamma v_n) = O. \end{cases} \quad (2.14)$$

System (2.13) (or (2.14)) is a rather general system involving several free parameters and thus encompasses simpler lattice systems as particular (or limiting) cases; some of them are already known.

- By rescaling the dependent variables and parameters as $u_n v_n = \alpha\beta u'_n v'_n$, $a = a'/\alpha$ and $b = b'/\beta$, omitting the prime and taking the limit $\alpha, \beta \rightarrow 0$, (2.13) reduces to the matrix generalization [36] (also see [16, 37] and references therein) of the Ablowitz–Ladik lattice [22]:

$$\begin{cases} u_{n,t} + a\delta(I - u_n v_n)u_{n-1} - b\gamma u_{n+1}(I - v_n u_n) - (c - d)u_n = O, \\ v_{n,t} + b\gamma(I - v_n u_n)v_{n-1} - a\delta v_{n+1}(I - u_n v_n) + (c - d)v_n = O. \end{cases}$$

This is a linear combination of the two elementary flows (2.4) and (2.5) and the trivial zeroth flow of the Ablowitz–Ladik hierarchy.

- By rescaling the dependent variables and parameter as $u_n v_n = \beta\gamma u'_n v'_n$ and $b = b'/\beta$, omitting the prime and taking the limit $\beta \rightarrow 0$, (2.14) reduces to the already known space-discretization of the Chen–Lee–Liu system [16, 17]:

$$\begin{cases} u_{n,t} - a\delta(\gamma u_n - \alpha u_{n-1})(I - v_n u_{n-1})^{-1} - \frac{b}{\alpha}(\gamma u_{n+1} - \alpha u_n)(\alpha I - \gamma v_n u_n) = O, \\ v_{n,t} - \frac{b}{\alpha}(\alpha I - \gamma v_n u_n)(\alpha v_n - \gamma v_{n-1}) - a\delta(I - v_{n+1} u_n)^{-1}(\alpha v_{n+1} - \gamma v_n) = O. \end{cases}$$

In the case of scalar u_n and v_n , this system was previously studied in [18–20]; in fact, it can be considered as a continuous-time analog of the fully discrete Chen–Lee–Liu system proposed by Date, Jimbo and Miwa [8], associated with an elementary auto-Bäcklund transformation for the continuous Chen–Lee–Liu hierarchy (cf. [21]). Unfortunately, this system does not admit a complex/Hermitian conjugation reduction between u_n and v_n , so it cannot provide a proper discretization of the Chen–Lee–Liu equation.

- In the special case of $\alpha = -\delta$ and $\beta = -\gamma$, (2.13) reduces to a nontrivial lattice system

$$\begin{cases} u_{n,t} - a\gamma(\gamma\delta I - u_n v_n)(\gamma^2 I + u_{n-1} v_n)^{-1}(\gamma u_n + \delta u_{n-1}) \\ \quad + b\delta(\gamma u_{n+1} + \delta u_n)(\delta^2 I + v_n u_{n+1})^{-1}(\gamma\delta I - v_n u_n) - (c-d)u_n = O, \\ v_{n,t} - b\delta(\gamma\delta I - v_n u_n)(\delta^2 I + v_{n-1} u_n)^{-1}(\delta v_n + \gamma v_{n-1}) \\ \quad + a\gamma(\delta v_{n+1} + \gamma v_n)(\gamma^2 I + u_n v_{n+1})^{-1}(\gamma\delta I - u_n v_n) + (c-d)v_n = O, \end{cases}$$

while the Lax pair given by (2.8) and (2.11) becomes trivial in the naive limit $\alpha \rightarrow -\delta$, $\beta \rightarrow -\gamma$. This lattice system can be derived from a Bäcklund–Darboux transformation for the continuous NLS hierarchy in the same manner as (2.13) is derived from a Bäcklund–Darboux transformation for the Ablowitz–Ladik hierarchy. By further setting $\gamma = \delta = \pm 1$, we obtain a new proper space-discretization of the matrix NLS system as well as the matrix modified KdV system. We will discuss it in a separate paper.

Not all of the four parameters α , β , γ and δ in (2.13) (or (2.14)) are essential; the number of independent parameters can be reduced by applying a point transformation of the form: $u_n = \mu\nu^n u'_n$, $v_n = \nu^{-n} v'_n$ with nonzero constants μ and ν . However, it then becomes difficult to consider limiting cases where one or more parameters tend to zero as mentioned above, so we prefer the “redundant” expression (2.13) (or (2.14)).

By setting $\gamma = \alpha$, $\delta = -\beta$ and $\alpha\beta = 1$, (2.14) provides a new integrable space-discretization of the Chen–Lee–Liu system:

$$\begin{cases} u_{n,t} + a(u_n - u_{n-1})(I - v_n u_{n-1})^{-1}(I + v_n u_n) \\ \quad - b(u_{n+1} - u_n)(I + v_n u_{n+1})^{-1}(I - v_n u_n) = O, \\ v_{n,t} - b(I - v_n u_n)(I + v_{n-1} u_n)^{-1}(v_n - v_{n-1}) \\ \quad + a(I + v_n u_n)(I - v_{n+1} u_n)^{-1}(v_{n+1} - v_n) = O. \end{cases} \quad (2.15)$$

System (2.15) with $b = -a^*$ admits the Hermitian conjugation reduction $v_n = iu_n^\dagger$; in the case of $a = b = i$, we obtain a proper space-discretization of the Chen–Lee–Liu equation [14] (see [44, 45, 60–62] for the matrix case):

$$\begin{aligned} iu_{n,t} + (u_{n+1} - u_n)(I + iu_n^\dagger u_{n+1})^{-1}(I - iu_n^\dagger u_n) \\ - (u_n - u_{n-1})(I - iu_n^\dagger u_{n-1})^{-1}(I + iu_n^\dagger u_n) = O. \end{aligned}$$

In the scalar case, this reads

$$iu_{n,t} + \frac{1 - i|u_n|^2}{1 + iu_n^* u_{n+1}} (u_{n+1} - u_n) - \frac{1 + i|u_n|^2}{1 - iu_n^* u_{n-1}} (u_n - u_{n-1}) = 0,$$

or equivalently,

$$iu_{n,t} + (u_{n+1} + u_{n-1} - 2u_n) - i \frac{(u_{n+1}^2 - u_n^2) u_n^*}{1 + iu_n^* u_{n+1}} - i \frac{(u_n^2 - u_{n-1}^2) u_n^*}{1 - iu_n^* u_{n-1}} = 0.$$

In addition, system (2.15) with $b = -a$ admits the matrix transpose reduction $v_n = u_n^T C$, where C is a constant skew-symmetric matrix [63]; in particular, in the case of $a = 1$ and $b = -1$, we obtain

$$\begin{aligned} u_{n,t} + (u_{n+1} - u_n) (I + u_n^T C u_{n+1})^{-1} (I - u_n^T C u_n) \\ + (u_n - u_{n-1}) (I - u_n^T C u_{n-1})^{-1} (I + u_n^T C u_n) = 0, \quad C^T = -C. \end{aligned}$$

In the scalar case, (2.13) is a Hamiltonian system with an ultralocal (but noncanonical) Poisson structure. Indeed, it can be expressed as $u_{n,t} = \{u_n, H\}$ and $v_{n,t} = \{v_n, H\}$, where the Hamiltonian and the Poisson brackets are given by

$$H = \sum_n \left[a \log \left(\frac{\beta\gamma - u_{n-1}v_n}{\gamma\delta - u_n v_n} \right) - b \log \left(\frac{\alpha\delta - u_{n+1}v_n}{\gamma\delta - u_n v_n} \right) + \frac{c - d}{\alpha\beta - \gamma\delta} \log \left(\frac{\alpha\beta - u_n v_n}{\gamma\delta - u_n v_n} \right) \right]$$

and

$$\{u_m, u_n\} = \{v_m, v_n\} = 0, \quad \{u_m, v_n\} = \delta_{mn} (\alpha\beta - u_n v_n) (\gamma\delta - u_n v_n),$$

respectively. Here, δ_{mn} is the Kronecker delta, which should not be confused with the free parameter δ . This Hamiltonian structure encompasses the already known Hamiltonian structures for the simpler lattice systems in the scalar case [18–20, 39, 64, 65]. It would be interesting to construct the corresponding classical r -matrix a la Sklyanin [66, 67].

3 Space-discrete massive Thirring model

In this section, we describe how to discretize one of the two independent variables of the massive Thirring model in light-cone (or characteristic) coordinates [33–35], which is the first negative flow of the Chen–Lee–Liu hierarchy [44, 48] (see [46, 47] for the model in laboratory coordinates). With a slight abuse of terminology, we will call it a “space-discretization” of the

massive Thirring model, although it would be more natural and appropriate to use it for the massive Thirring model in laboratory coordinates.

The generalized binary Bäcklund–Darboux transformation (2.6) with suitably defined intermediate potentials u_m and v_m can preserve the Ablowitz–Ladik spectral problem (2.1) form-invariant, as well as an infinite set of isospectral time-evolution equations such as (2.2) and (2.3). Thus, the temporal Lax matrix M_n given by (2.11) with (2.12) is not the only possible choice; it can be replaced with any M_n -matrix corresponding to a flow of the Ablowitz–Ladik hierarchy, or equivalently, the continuous Chen–Lee–Liu hierarchy. To obtain a space-discrete analog of the massive Thirring model, we suppress the lattice index m for the Ablowitz–Ladik hierarchy and consider the isospectral time-evolution equation (2.3) written in the form:

$$\begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}_t = \begin{bmatrix} O & \frac{1}{\zeta} Q_n \\ \frac{1}{\zeta} R_n & \frac{1}{\zeta^2} I - R_n Q_n \end{bmatrix} \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}.$$

Then, substituting the Lax pair given by (2.8) and

$$M_n = \begin{bmatrix} O & \frac{1}{\zeta} Q_n \\ \frac{1}{\zeta} R_n & \frac{1}{\zeta^2} I - R_n Q_n \end{bmatrix}$$

into the zero-curvature equation (2.10), we obtain a set of six equations. A direct calculation shows that only four of them are independent, which can be presented as

$$\left\{ \begin{array}{l} \beta\delta(\gamma Q_{n+1} - \alpha Q_n) + (\alpha\beta - \gamma\delta)u_n - u_n v_n(\beta Q_{n+1} - \delta Q_n) = O, \\ \beta\delta(\alpha R_{n+1} - \gamma R_n) - (\alpha\beta - \gamma\delta)v_n - (\delta R_{n+1} - \beta R_n)u_n v_n = O, \\ (\alpha\beta - \gamma\delta)u_{n,t} + \alpha\gamma(\beta Q_{n+1} - \delta Q_n) + u_n(\alpha R_{n+1} - \gamma R_n)u_n \\ \quad - u_n(\alpha\beta R_{n+1}Q_{n+1} - \gamma\delta R_n Q_n) = O, \\ (\alpha\beta - \gamma\delta)v_{n,t} + \alpha\gamma(\delta R_{n+1} - \beta R_n) + v_n(\gamma Q_{n+1} - \alpha Q_n)v_n \\ \quad - (\gamma\delta R_{n+1}Q_{n+1} - \alpha\beta R_n Q_n)v_n = O. \end{array} \right. \quad (3.1)$$

By setting $\gamma = \alpha = i$ and $\delta = -\beta = -2i/\Delta$ where $\Delta \in \mathbb{R}$ is a lattice param-

eter, (3.1) reads

$$\left\{ \begin{array}{l} \frac{2}{\Delta} (Q_{n+1} - Q_n) + 2iu_n - u_n v_n (Q_{n+1} + Q_n) = O, \\ \frac{2}{\Delta} (R_{n+1} - R_n) - 2iv_n + (R_{n+1} + R_n) u_n v_n = O, \\ 2u_{n,t} + i(Q_{n+1} + Q_n) - \frac{i\Delta}{2} u_n (R_{n+1} - R_n) u_n \\ \quad - u_n (R_{n+1} Q_{n+1} + R_n Q_n) = O, \\ 2v_{n,t} - i(R_{n+1} + R_n) - \frac{i\Delta}{2} v_n (Q_{n+1} - Q_n) v_n \\ \quad + (R_{n+1} Q_{n+1} + R_n Q_n) v_n = O. \end{array} \right. \quad (3.2)$$

Note that the set of coefficients in (3.2) can be changed by rescaling the variables. System (3.2) admits the Hermitian conjugation reduction $R_n = iQ_n^\dagger$, $v_n = iu_n^\dagger$, so we obtain a proper space-discretization of the massive Thirring model as

$$\left\{ \begin{array}{l} \frac{2}{\Delta} (Q_{n+1} - Q_n) + 2iu_n - iu_n u_n^\dagger (Q_{n+1} + Q_n) = O, \\ 2u_{n,t} + i(Q_{n+1} + Q_n) + \frac{\Delta}{2} u_n (Q_{n+1}^\dagger - Q_n^\dagger) u_n \\ \quad - iu_n (Q_{n+1}^\dagger Q_{n+1} + Q_n^\dagger Q_n) = O. \end{array} \right. \quad (3.3)$$

In the continuous space limit, (3.3) indeed reduces to the massive Thirring model in light-cone coordinates (see (3.48) in [44]):

$$\left\{ \begin{array}{l} Q_x + iu - iuu^\dagger Q = O, \\ u_t + iQ - iuQ^\dagger Q = O. \end{array} \right.$$

We remark that another lattice version of the massive Thirring model in light-cone coordinates was studied in [68].

4 Concluding remarks

In this paper, we have developed an effective approach for generating new integrable lattice systems from Bäcklund–Darboux transformations for known integrable systems. The idea to interpret a Bäcklund–Darboux transformation as a discrete spectral problem is already well-known (see, *e.g.*, [6–8]);

the main new feature of our approach is to consider the intermediate potentials appearing in a binary Bäcklund–Darboux transformation as new dependent variables in the discrete spectral problem, up to a rescaling of the variables. As the name implies, the binary Bäcklund–Darboux transformation is equivalent to the composition of two elementary Bäcklund–Darboux transformations in either order of the composition [5, 69, 70]. In the example of the Ablowitz–Ladik hierarchy, u_m and v_m in (2.6) appear in the decomposition of the binary Bäcklund–Darboux transformation into two permutable elementary Bäcklund–Darboux transformations as [50]

$$(Q_m, R_m) \rightarrow (u_m, *) \rightarrow (\tilde{Q}_m, \tilde{R}_m)$$

and

$$(Q_m, R_m) \rightarrow (*, v_m) \rightarrow (\tilde{Q}_m, \tilde{R}_m)$$

up to a rescaling and a space translation, so they are indeed the intermediate potentials; a similar decomposition holds true for the continuous Chen–Lee–Liu hierarchy if one considers the pair of variables (Q_m, R_{m-1}) or (Q_{m-1}, R_m) (cf. (2.2) or (2.3)). Then, we associate the new discrete spectral problem with a suitable isospectral time-evolution equation on a case-by case consideration and obtain an evolutionary lattice system from the compatibility condition called the zero-curvature equation. We illustrated this approach by deriving the general lattice system (2.14), which involves the arbitrary parameters α , β , γ and δ and includes the space-discrete Chen–Lee–Liu system (2.15) as a special case; the derivation of negative flows of the integrable hierarchy is easier and can be performed in a more systematic manner as is illustrated in section 3. This approach is quite useful for obtaining proper discretizations of integrable systems that admit the complex/Hermitian conjugation reduction between the two dependent variables, such as the NLS system, derivative NLS systems and their matrix generalizations; this is because the intermediate potentials generally inherit the internal symmetries of the original continuous system, which guarantees the feasibility of such a reduction.

In the continuous case, by applying a nonlocal transformation of dependent variables [41, 48, 71, 72], we can transform the Chen–Lee–Liu equation to other derivative NLS equations such as the Kaup–Newell equation [71] and the Ablowitz–Ramani–Segur (Gerdjikov–Ivanov) equation [73, 74]; the same transformation applies to other flows of the integrable hierarchy including the massive Thirring model [33, 35, 48, 57] (also see [56]). Note that the nonlocal quantity used in this transformation is given by the first (or second) component of the linear eigenfunction of the Lax-pair representation with the spectral parameter set equal to zero. In a similar manner, we can obtain new proper space-discretizations of other derivative NLS systems from

the space-discrete Chen–Lee–Liu system (2.15) by applying a discrete analog of the nonlocal transformation. Indeed, (2.7) with (2.8b) implies that the quantity $Z_n := (-\delta)^{-n} e^{\xi t} \lim_{\zeta \rightarrow 0} \zeta^n \Psi_{1,n}$ satisfies the following relation for any constant ξ :

$$Z_{n+1} = (\alpha\beta I - u_n v_n) (-\gamma\delta I + u_n v_n)^{-1} Z_n.$$

The time derivative of Z_n for a suitably chosen ξ can be obtained from (2.9) with (2.11) and (2.12) as

$$Z_{n,t} = (\alpha\beta - \gamma\delta) [a\beta\gamma(\beta\gamma I - u_{n-1}v_n)^{-1} + b\alpha\delta(\alpha\delta I - u_n v_{n-1})^{-1}] Z_n.$$

Thus, the nonlocal transformation

$$q_n := Z_n^{-1} u_n, \quad r_n := v_n Z_n$$

or its one-parameter generalization in the scalar case

$$q_n := Z_n^{-k} u_n, \quad r_n := v_n Z_n^k$$

can be applied to (2.14); by setting $\gamma = \alpha$, $\delta = -\beta$ and $\alpha\beta = 1$, (2.14) reduces to (2.15) and we obtain proper space-discretizations of other derivative NLS systems.

In the original formulation of Bäcklund–Darboux transformations [5–7, 30–32], the intermediate potentials such as u_m and v_m in (2.6) can be written explicitly in terms of the linear eigenfunctions at some fixed values of the spectral parameter ζ . Then, Bäcklund–Darboux transformations can be applied iteratively to obtain a sequence of new solutions of a nonlinear integrable system from its seed solution and the associated linear eigenfunctions; the final result does not depend on the order of applications. That is, Bäcklund–Darboux transformations with (generally) different values of the Bäcklund parameters are mutually commutative as long as boundary conditions are fixed appropriately; this fact can be understood intuitively by identifying Bäcklund–Darboux transformations as discrete-time flows that belong to the same integrable hierarchy. Without taking into account how to express the intermediate potentials in terms of the linear eigenfunctions of the Lax pair, the permutability condition for Bäcklund–Darboux transformations results in a matrix re-factorization problem (see, *e.g.*, [75–77]). In the case considered in this paper, it reads [7, 76–79]

$$L(\mathcal{U}', \mathcal{V}'; \mu, \nu, \xi, \eta) L(u, v; \alpha, \beta, \gamma, \delta) = L(u', v'; \alpha, \beta, \gamma, \delta) L(\mathcal{U}, \mathcal{V}; \mu, \nu, \xi, \eta), \quad (4.1)$$

where the matrix $L(u, v; \alpha, \beta, \gamma, \delta)$ is given by (2.8) with the unnecessary lattice index n removed and the dependence on the spectral parameter ζ suppressed. Considering the matrix inverse of L , we can rewrite (4.1) as [78, 79]

$$L(u, v; \gamma, \delta, \alpha, \beta)L(\mathcal{U}', \mathcal{V}'; \xi, \eta, \mu, \nu) = L(\mathcal{U}, \mathcal{V}; \xi, \eta, \mu, \nu)L(u', v'; \gamma, \delta, \alpha, \beta). \quad (4.2)$$

Equation (4.1) or (4.2) can be naturally represented using an elementary quadrilateral and defines a parameter-dependent Yang–Baxter map

$$(u, v; \mathcal{U}', \mathcal{V}') \mapsto (u', v'; \mathcal{U}, \mathcal{V}),$$

wherein the fields are assigned to the edges of the quadrilateral, instead of the vertices [77–80]. The Lax (or zero-curvature) representation for the Yang–Baxter map is given by (4.2) [76–79]. It is not evident from the explicit form of the Lax matrix in (2.8) that (4.2) can be solved uniquely to provide the Yang–Baxter map, but the map itself can, in principle, be constructed using the definition of the intermediate potentials u and v in terms of the linear eigenfunctions.

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